ARPM Certificate

Level 1

Example with solutions

Solution 1 True or false? Justify your answer

i) *[3 points]* The security market line equation holds only under the assumption of a normal market.

False. The security market line equation is an identity that holds true for any distribution of the market, under the fundamental axioms of linear pricing theory.

ii) *[3 points]* In a linear pricing framework, the payoff of a financial instrument at a future horizon is regarded as a random variable, and the current value is given by the corresponding expectation.

False. The current value is given by the expectation of the product of the payoff and a stochastic discount factor.

iii) *[3 points]* A reasonable guess for the price of an equity share tomorrow is the sample average over a time series of past prices.

False. The price of a stock is not i.i.d. across time. Therefore, the sample mean of past prices is not informative about the future price.

iv) *[3 points]* The yield to maturity $Y_t(\tau)$ at a given time t is the (rescaled) compounded return of a zero coupon bond from time t to the maturity date.

True. The yield to maturity is defined as

$$Y_t(\tau) \equiv -\frac{1}{\tau} \ln V_t^{zcb}(t+\tau) = \frac{1}{\tau} \ln\left(\frac{v_{t_{end}}^{zcb}(t_{end})}{V_t^{zcb}(t_{end})}\right),$$

where we used the condition $v_{t_{end}}^{zcb}(t_{end}) \equiv 1$.

v) *[3 points]* The 10-year yield to maturity of the zero swap curve, sampled over daily time steps, is approximately independent and identically distributed through time.

False. The 10-year yield to maturity of the zero swap curve, sampled over daily time steps, is approximately a random walk. Hence, its *increments* are approximately i.i.d. across time.

vi) *[3 points]* If the 10-year yield to maturity follows a stationary Ornstein-Uhlenbeck process, the square-root rule does not apply, i.e. the propagation of the standard deviation is not proportional to the square-root of the time to the horizon.

True. The standard deviation is approximately proportional to the square-root of the time to the horizon *only* in the short run. Over longer horizon, the standard deviation converges to the unconditional standard deviation.

vii) *[3 points]* If the yield curve follows a multivariate Ornstein-Uhlenbeck (MVOU) process, the conditional distribution of the yields to maturity at any future horizon is multivariate normal.

True. The conditional distribution of the yields to maturity at the horizon $t_{hor} = t_{now} + \Delta t$ is

$$\boldsymbol{Y}_{t_{now}+\Delta t}|\boldsymbol{i}_{t_{now}} \sim N(e^{-\boldsymbol{\theta}\Delta t}\boldsymbol{y}_{t_{now}} + \boldsymbol{\mu}_{\Delta t}, \boldsymbol{\sigma}^2_{\Delta t}),$$

where

$$\boldsymbol{\mu}_{\Delta t} = (\mathbb{I}_{\bar{n}} - e^{-\boldsymbol{\theta}\Delta t})\boldsymbol{\theta}^{-1}\boldsymbol{\mu}.$$

and

$$\boldsymbol{\sigma}^2_{\Delta t} = vec^{-1}[(\boldsymbol{\theta} \oplus \boldsymbol{\theta})^{-1}(\mathbb{I}_{\bar{n}^2} - e^{-(\boldsymbol{\theta}\oplus\boldsymbol{\theta})\Delta t})vec(\boldsymbol{\sigma}^2)].$$

viii) *[3 points]* It is possible to build a principal-component linear factor model on $\bar{n} \equiv 6$ target variables using $\bar{k} = 9$ factors.

False. Principal-component linear factor models are defined only for a number $\bar{k}$ of factors $\boldsymbol{Z}^{PC}$ lower than the number $\bar{n}$ of target variables $\boldsymbol{X}$, $\bar{k} \leq \bar{n}$. This is the reason why principal-component linear factor models are particularly useful to perform *dimension reduction*.

ix) *[3 points]* If the supervised probabilistic prediction for a univariate output X and input Z is a conditional Bernoulli distribution, $X|z \sim Bernoulli(p(z))$, then the conditional probability of the positive outcome $p(z)$ is a supervised point prediction.

True. If $X|z \sim Bernoulli(p(z))$ is the supervised probabilistic prediction and we consider the corresponding point prediction using the expectation as location functional we obtain

$$\mathbb{E}\{X|z\} = 0 \times (1 - p(z)) + 1 \times p(z) = p(z).$$

x) *[3 points]* Regression linear factor models are unsupervised autoencoders.

False. Regression linear factor models are supervised linear point predictors.

Solution 2 Scenario-probability distributions, historical estimation, location-dispersion ellipsoid

Consider a bivariate time series $\{\boldsymbol{\epsilon}_t\}_{t=1}^{\bar{t}}$ with $\bar{t} = 3$ observations

$$\{\underset{\boldsymbol{\epsilon}_1}{\underset{\uparrow}{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}}, \quad \underset{\boldsymbol{\epsilon}_2}{\underset{\uparrow}{\begin{pmatrix} 0 \\ -1 \end{pmatrix}}}, \quad \underset{\boldsymbol{\epsilon}_3}{\underset{\uparrow}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}}\}. \tag{1}$$

i) *[3 points]* Compute the historical expectation $\hat{\boldsymbol{m}}^{Hist}_{\boldsymbol{\varepsilon}}$ and covariance $\hat{\boldsymbol{s}}^{2Hist}_{\boldsymbol{\varepsilon}}$.

The historical expectation is

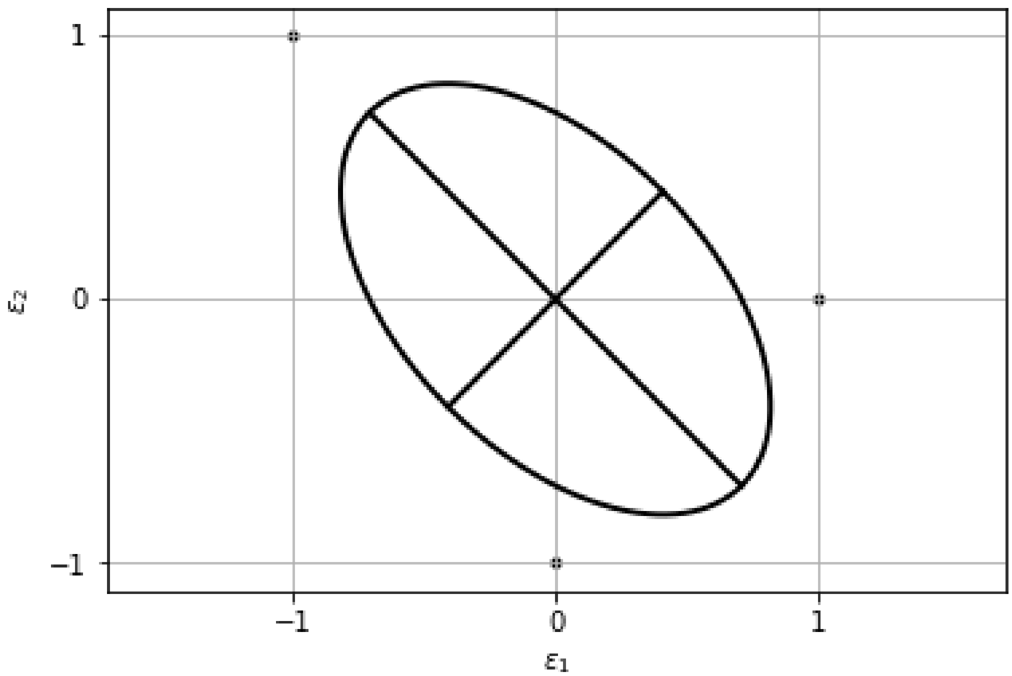
$$\hat{\boldsymbol{m}}_{\boldsymbol{\varepsilon}}^{Hist} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and the historical covariance is

$$\hat{\boldsymbol{s}}_{\boldsymbol{\varepsilon}}^{2Hist} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

ii) *[6 points]* Draw the scatter plot of (1) and sketch the location-dispersion ellipsoid $\mathfrak{E}(\hat{\boldsymbol{m}}_{\boldsymbol{\varepsilon}}^{Hist}, \hat{\boldsymbol{s}}_{\boldsymbol{\varepsilon}}^{2Hist})$ (center and principal axes).

- ARPM Certificate Exams and Solutions/Example - Level 1/ellipse.png



The center of the ellipse is $\hat{\boldsymbol{m}}_{\boldsymbol{\varepsilon}}^{Hist} = (0,0)'$.
The directions of the principal axes are given by the eigenvectors of $\hat{\boldsymbol{s}}_{\boldsymbol{\varepsilon}}^{2Hist}$, which are

$$\boldsymbol{e}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \boldsymbol{e}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The length of the principal axes are the square root λ_1 and λ_2 of the eigenvalues of $\hat{\boldsymbol{s}}_{\boldsymbol{\varepsilon}}^{2Hist}$, which are

$$\lambda_1^2 = 1, \quad \lambda_2^2 = \frac{1}{3}.$$

iii) *[3 points]* Compute the historical with flexible probabilities expectation $\hat{\boldsymbol{m}}_{\boldsymbol{\varepsilon}}^{HFP}$ and covariance $\hat{\boldsymbol{s}}_{\boldsymbol{\varepsilon}}^{2HFP}$, given the probabilities

$$p_1 = 20\%, \quad p_2 = 20\%, \quad p_3 = 60\%.$$

The historical with flexible probabilities expectation is

$$\hat{\boldsymbol{m}}_{\boldsymbol{\varepsilon}}^{HFP} = \left(\begin{smallmatrix} 0.4 \\ 0 \end{smallmatrix}\right)$$

and the historical with flexible probabilities covariance is

$$\hat{\boldsymbol{s}}_{\boldsymbol{\varepsilon}}^{2HFP} = \left(\begin{smallmatrix} 0.64 & -0.2 \\ -0.2 & 0.4 \end{smallmatrix}\right).$$

iv) *[3 points]* Compute the scenario-probability distribution of the transformed random variable $\boldsymbol{\varepsilon}^2 - \sin(\frac{\pi}{2}\boldsymbol{\varepsilon})$.

The random variable $\boldsymbol{\varepsilon}^2 - \sin(\frac{\pi}{2}\boldsymbol{\varepsilon})$ has scenario-probability distribution

$$\boldsymbol{\varepsilon}^2 - \sin(\frac{\pi}{2}\boldsymbol{\varepsilon}) \sim \{\left(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right), 20\%, \quad \left(\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right), 20\%, \quad \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), 60\%, \}.$$

Solution 3 The Checklist applied to stocks: steps 1 to 5 with copula-marginal approach *[Tot. 30 points]*

Consider as risk drivers the log-values of the 500 stocks in the S&P 500.
Assume that each risk driver follows a GARCH(1,1) process over a daily time step.
Assume that the GARCH(1,1) shocks are independent and identically distributed through time. Model their marginal distributions via the respective historical distributions and the (static) copula as a t variable with $\nu = 5$ degrees of freedom.
Summarize in detail, using pseudo-code, the steps that you would apply to find the joint distribution of the P&L;s of the 500 stocks from today to tomorrow.

Step 1. Risk drivers identification *[2 points]*

for $d = 1, \ldots, 500$
Risk drivers $\{x_{d,t} \equiv \ln v_{d,t}^{stock}\}_{t=0,\ldots,\bar{t}}$

Step 2. Quest for invariance *[8 points]*

for $d = 1, \ldots, 500$

2a) Increments $\{\Delta x_{d,t} \equiv x_{d,t} - x_{d,t-1}\}_{t=1}^{\bar{t}}$

2b) Initialize variance (forward smoothing) $\hat{\sigma}^2_{d,0} \leftarrow \frac{\sum_{s=1}^{\bar{t}} e^{-\frac{\ln(2)}{\tau_{HL}}|1-s|} \Delta x_{d,s}}{\sum_{s=1}^{\bar{t}} e^{-\frac{\ln(2)}{\tau_{HL}}|1-s|}}$

2c) Fit GARCH(1,1) (max. likelihood) $(\hat{a}_d, \hat{b}_d, \hat{c}_d, \hat{\mu}_d) \leftarrow \texttt{fit\_garch}(\{\Delta x_{d,t}\}_{t=1}^{\bar{t}}, \hat{\sigma}^2_{d,0})$

2d) Extract invariants

for $t = 1, \ldots, \bar{t}$

Volatility $\hat{\sigma}^2_{d,t} \leftarrow \hat{c}_d + \hat{b}_d \hat{\sigma}^2_{d,t-1} + \hat{a}_d \left(\Delta x_{d,t-1} - \hat{\mu}_d\right)^2$

Invariant $\epsilon_{d,t} \leftarrow (\Delta x_{d,t} - \hat{\mu}_d)/\hat{\sigma}_t$

Step 3. Estimation *[8 points]*

for $d = 1, \ldots, 500$

3a) Marginal cdf $F^{Hist}_{\varepsilon_d}(x) \equiv \frac{1}{\bar{t}} \sum_{t=1}^{\bar{t}} 1_{\epsilon_{d,t} \leq x}$

3b) Grade $u_{d,t} \leftarrow F^{Hist}_{\varepsilon_d}(\epsilon_{d,t})$

3c) Standardized invariants $\tilde{\epsilon}_{d,t} \equiv \Phi_5^{-1}(u_{d,t})$

3d) Fit t copula via ML $(\sim, \boldsymbol{\rho}^2) \leftarrow \texttt{fit\_locdisp\_mlfp}(\{\tilde{\boldsymbol{\epsilon}}_t, \frac{1}{\bar{t}}\}_{t=1}^{\bar{t}}, \nu = 5)$

Step 4. Projection *[8 points]*

4a) Next-step standardized invariants $\tilde{\boldsymbol{\epsilon}}^{(j)} \leftarrow \texttt{simulate\_t}(\mathbf{0}, \boldsymbol{\rho}^2, 5, \bar{j})$

for $d = 1, \ldots, 500$

4b) Grades $\{u_d^{(j)} \leftarrow \Phi_5(\tilde{\epsilon}_d^{(j)})\}_{j=1}^{\bar{j}}$

4c) Next-step invariants $\{\epsilon_d^{(j)} \leftarrow (F^{Hist}_{\varepsilon_d})^{-1}(u_d^{(j)})\}_{j=1}^{\bar{j}}$

4d) Next-step volatility $\hat{\sigma}_{d,t_{now}+1} \leftarrow \hat{c}_d + \hat{b}_d \hat{\sigma}^2_{d,t_{now}} + \hat{a}_d \left(\Delta x_{d,t_{now}} - \hat{\mu}_d\right)^2$

4e) Next-step risk drivers $\{x_d^{(j)} \leftarrow x_{d,t_{now}} + \hat{\mu}_d + \hat{\sigma}_{d,t_{now}+1} \epsilon_d^{(j)}\}_{j=1}^{\bar{j}}$

Step 5. Pricing *[4 points]*

for $d = 1, \ldots, 500$

5a) Value at the horizon $\{v_d^{(j)} \leftarrow \exp(x_d^{(j)})\}_{j=1}^{\bar{j}}$

5b) P&L $\{\pi_d^{(j)} \leftarrow v_d^{(j)} - v_{d,t_{now}}\}_{j=1}^{\bar{j}}$

5c) Joint P&L distribution $\boldsymbol{\Pi}_{t_{now}\rightarrow t_{now}+1} \sim \{\boldsymbol{\pi}^{(j)} \equiv \begin{pmatrix} \pi_1^{(j)} \\ \cdot \\ \pi_d^{(j)} \\ \cdot \\ \pi_{500}^{(j)} \end{pmatrix}, p^{(j)} = \frac{1}{\bar{j}}\}_{j=1}^{\bar{j}}$

Solution 4 Non-parametric cross-sectional linear factor models *[Tot. 25 points]*

Consider a cross-sectional linear factor model for a $(\bar{n} = 2)$-dimensional target variable $\boldsymbol{X}$ with $\bar{k} = 1$ factor Z^{CS}

$$\boldsymbol{X} = \boldsymbol{\alpha} + \boldsymbol{\beta} Z^{CS} + \boldsymbol{U}. \tag{2}$$

Recall that the cross-sectional linear factor model (2) solves the following problem

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}, Z^{CS}) \equiv \underset{(\boldsymbol{a},\boldsymbol{b},F)\in\mathcal{C}}{\operatorname{argmax}}\, \mathcal{R}^2_{\boldsymbol{\sigma}^2}\{\boldsymbol{a} + \boldsymbol{b}F||\boldsymbol{X}\} \tag{3}$$

where the objective is the multivariate r-squared

$$\mathcal{R}^2_{\boldsymbol{\sigma}^2}\{\boldsymbol{Y}||\boldsymbol{X}\} = 1 - \frac{\mathbb{E}\{||\boldsymbol{\sigma}^{-1}(\boldsymbol{Y} - \boldsymbol{X})||^2\}}{\operatorname{tr}(\mathbb{C}v\{\boldsymbol{\sigma}^{-1}\boldsymbol{X}\})} \tag{4}$$

and the constraints read

$$\mathcal{C} : \begin{cases} \boldsymbol{b} = \boldsymbol{\beta} \\ F = \boldsymbol{c}\boldsymbol{X} \\ \boldsymbol{a} = (\mathbb{I}_{\bar{n}} - \boldsymbol{\beta}\boldsymbol{c})\mathbb{E}\{\boldsymbol{X}\} \end{cases} . \tag{5}$$

Assume that the target variables have the following scenario-probability distribution

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), 25\%, \quad \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right), 25\%, \quad \left(\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right), 25\%, \quad \left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right), 25\%\},$$

that the observable loadings read

$$\boldsymbol{\beta} \equiv (1, 0)',$$

and the scale matrix specifying the r-squared (4) reads

$$\boldsymbol{\sigma}^2 \equiv \left(\begin{smallmatrix} 1.25 & 0.5 \\ 0.5 & 0.5 \end{smallmatrix}\right). \tag{6}$$

i) *[3 points]* Compute the shift parameter $\boldsymbol{\alpha} \equiv (\alpha_1, \alpha_2)'$ and the factor-construction parameter $\boldsymbol{\gamma} \equiv (\gamma_1, \gamma_2)$, which correspond to the optimal $\boldsymbol{a}$ and $\boldsymbol{c}$ in problem (3)-(5), respectively.

The shift parameter is

$$\boldsymbol{\alpha} \equiv \left(\begin{smallmatrix} \alpha_1 \\ \alpha_2 \end{smallmatrix} \right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right).$$

and the factor-construction parameter is

$$\boldsymbol{\gamma} \equiv (\gamma_1, \gamma_2) = (1, -1).$$

ii) *[5 points]* Compute the joint distribution of residuals and factor $(U_1, U_2, Z^{CS})'$.

The residuals and factor are jointly distributed according to the following scenario-probability distribution

$$\begin{pmatrix} U_1 \\ U_2 \\ Z^{CS} \end{pmatrix} \sim \{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, 25\%, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, 25\%, \quad \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, 25\%, \quad \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, 25\% \}.$$

iii) *[2 points]* Show that this cross-sectional linear factor model is systematic, but *not* idiosyncratc.

The joint covariance of residuals and factor reads

$$\mathbb{C}v\{ \begin{pmatrix} U_1 \\ U_2 \\ Z^{CS} \end{pmatrix} \} = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 0.75 \end{pmatrix}.$$

Hence the cross-sectional linear factor model is systematic because the residuals $\boldsymbol{U} \equiv (U_1, U_2)'$ and the factor Z^{CS} are uncorrelated; but is *not* idiosyncratic, since the residuals $\boldsymbol{U} \equiv (U_1, U_2)'$ are correlated among each other.

iv) *[5 points]* Compute the optimal r-squared $\mathcal{R}^2_{\boldsymbol{\sigma}^2}\{\boldsymbol{\alpha} + \boldsymbol{\beta} Z^{CS} || \boldsymbol{X}\}$.

The optimal r-squared reads

$$\mathcal{R}^2_{\boldsymbol{\sigma}^2}\{\boldsymbol{\alpha} + \boldsymbol{\beta} Z^{CS} || \boldsymbol{X}\} = 0.5.$$

v) *[5 points]* Suppose now arbitrary loadings $\boldsymbol{\beta} \equiv (\beta_1, \beta_2)'$ and scale matrix as in (6). Explain why this cross-sectional linear factor model is systematic, regardless of the choice of $\boldsymbol{\beta}$.

The scale matrix is exactly the joint covariance of the target variables

$$\boldsymbol{\sigma}^2 \equiv \left(\begin{smallmatrix} 1.25 & 0.5 \\ 0.5 & 0.5 \end{smallmatrix} \right) = \mathbb{C}v\{ \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \}, \tag{7}$$

which represents the natural scale specification of cross-sectional models. Under this specification, independently of the choice of $\boldsymbol{\beta}$, the regression loadings stemming from cross-sectional factor Z^{CS} coincide always with the observable loadings $\boldsymbol{\beta}$. Therefore, as in the regression case, the cross-sectional model is always systematic because the residuals and the factors are never correlated.

vi) *[5 points]* Suppose arbitrary loadings $\boldsymbol{\beta} \equiv (\beta_1, \beta_2)'$ and scale matrix as in (6). Explain why the optimal r-squared reads $\mathcal{R}^2_{\boldsymbol{\sigma}^2}\{\boldsymbol{\alpha} + \boldsymbol{\beta} Z^{CS} || \boldsymbol{X}\} = 0.5$, regardless of the choice of $\boldsymbol{\beta}$.

According to the optimal scale specification (7), the r-squared $\mathcal{R}^2_{\boldsymbol{\sigma}^2}\{\boldsymbol{\alpha} + \boldsymbol{\beta} Z^{CS} || \boldsymbol{X}\}$ is always given by the number of factors divided by the number of the target variables, i.e.

$$\mathcal{R}^2_{\boldsymbol{\sigma}^2}\{\boldsymbol{X}^{CS} || \boldsymbol{X}\} = \frac{\bar{k}}{\bar{n}} = \frac{1}{2} = 0.5.$$